MATH 347: FUNDAMENTAL MATHEMATICS, FALL 2015

PRACTICE PROBLEMS FOR MITERM 3

1. Write the number $1043_{(5)}$ in 4-ary representation.

2. Let D be a set and let $f, g : D \to \mathbb{R}$ be bounded functions such that $\forall x \in D$, $f(x) \leq g(x)$. For each of the following statements, either prove it or give a counter-example.

(a) $\sup f(D) \leq \inf g(D)$.

(b) $\sup f(D) \leq \sup g(D)$.

(c) $\inf f(D) \leq \inf g(D)$.

3. Prove that for any sets $A, B \subseteq \mathbb{R}$ that are bounded above, $\sup(A \cup B) = \max\{\sup A, \sup B\}$.

Before continuing further, let's review the definition of limit.

Definition 1. Let $P(n)$ be a mathematical statement for every $n \in \mathbb{N}$. We say that **eventually** $P(n)$ holds if there is (an event) $N \in \mathbb{N}$ such that for every (moment) $n \geq N$, $P(n)$ holds.

Definition 2. We say that $L \in \mathbb{R}$ is a *limit* of a sequence $(x_n)_n$, and write $\lim_{n\to\infty} x_n = L$ or $x_n \to L$, if for every (measure of closeness) $\varepsilon > 0$, **eventually** $|x_n - L| < \varepsilon$ (i.e. x_n is within less than ε distance of L).

Rewriting the last definition without using the term **eventually**, we get the following (somewhat dry and hard to comprehend) reformulation:

Definition 2′. We say that $L \in \mathbb{R}$ is a *limit* of a sequence $(x_n)_n$ if

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \geq N \ |x_n - L| < \varepsilon.$$

It is also worth noting that the condition $|x_n - L| < \varepsilon$ can be written in various (equivalent) ways, such as:

(i) $-\varepsilon < x_n - L < \varepsilon$

(ii) $-\varepsilon < L - x_n < \varepsilon$

(iii) $L - \varepsilon < x_n < L + \varepsilon$

(iv) $x_n \in (L - \varepsilon, L + \varepsilon)$

(v) $x_n \in B(L, \varepsilon)$, where $B(L, \varepsilon)$ denotes the "open ball around L of radius ε", which simply means $B(L, \varepsilon) := (L - \varepsilon, L + \varepsilon)$.

5. Let $P(n)$ be a mathematical statement for every $n \in \mathbb{N}$. Write down explicitly the negation of the statement "**eventually** $P(n)$ holds".

6. Let $n_0 \in \mathbb{N}$. For a sequence $(x_n)_n$, let $(x_n)_{n \geq n_0}$ denote the sequence obtained from $(x_n)_n$ by deleting the first $n_0 - 1$ terms, i.e. $(x_{n_0}, x_{n_0+1}, x_{n_0+2}, ...)$. Prove that $(x_n)_n$ converges to L if and only if $(x_n)_{n \geq n_0}$ converges to L. In other words, the first finitely many terms don't affect the convergence of the sequence.

7. Suppose that $x_n \to L$ and $L > 7$. Prove that **eventually** $x_n > 7$.

8. For each of the following statements, determine whether they are true or false, and prove your answers.

(a) If a sequence is bounded, it has a limit.

(b) The sequence $(0, 1, 0, 1, ...)$ diverges.

(c) $\lim_{n\to\infty} \frac{(-1)^n n}{n+1} = -1$.

(d) If a sequence is monotone, it has a limit.

(e) If $(x_n \cdot y_n)_n$ converges, then at least one of $(x_n)_n$ and $(y_n)_n$ converges.

(f) If a bounded sequence $(x_n)_n$ is increasing, then it converges to $\sup\{x_n : n \in \mathbb{N}\}$.

9. Do Problems 1, 2(b) and 3 of HW10. If you have time, also do 2(a) and 4.